On the Traces of Hadamard and Kronecker Products of Matrices

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Abstract. In this paper we investigated traces of Hadamard and Kronecker products of matrices and obtained some inequalities for traces of products of matrices.

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1. Introduction

The initial studies about algebraic and analytic properties of Hadamard product are done by Schur in 1911. Because of the originality of this study, hadamard product is called Schur product. Horn gave a widespread information about Hadamard product in 1990 [1]. Marcus and Moyls [2] dwelled upon Hadamard product to be a principal submatrix of Kronecker product, also Visick [3] showed that $A \circ B = P_n^T (A \otimes B) P_n$, where $A$ and $B$ are arbitrary $n \times n$ matrices and $P$ is an $n^2 \times n$ selection matrix such that $P^T P = I$. Also, some studies, related with traces and matrix products, can be seen in [4-10].

Let $A = (a_{ij})$ be $n \times n$ matrix. Then the trace of $A$, denoted by $tr(A)$ is the sum of the major diagonal elements of $A$, that is

$$tr(A) = \sum_{i=1}^{n} a_{ii}.$$
The Hadamard product of two \( m \times n \) matrices \( A = (a_{ij}) \) and \( B = (b_{ij}) \) is defined as
\[
A \circ B = (a_{ij}b_{ij})_{m \times n}.
\]
If \( A = (a_{ij}) \) is an \( m \times n \) matrix and \( B = (b_{ij}) \) is a \( p \times q \) matrix, the Kronecker product \( A \otimes B \) is the \( mp \times nq \) matrix
\[
A \otimes B = 
\begin{pmatrix}
a_{11}B & a_{12}B & \cdots & a_{1n}B \\
a_{21}B & a_{22}B & \cdots & a_{2n}B \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1}B & a_{m2}B & \cdots & a_{mn}B
\end{pmatrix}.
\]

**Lemma 1.** For \( \forall i \in \mathbb{N}, \forall a_i \geq 0, \)
\[
(1) \quad \frac{1}{n} \left( \sum_{i=1}^{n} a_i \right) \geq \left( \prod_{i=1}^{n} a_i \right)^{1/n}.
\]

**Lemma 2.** Let \( A \) and \( B \) be arbitrary \( n \times n \) matrices. Then
\[
(2) \quad tr(A \otimes B) = tr(A)tr(B).
\]

**Proposition 1.** For \( \forall a, b \in \mathbb{R}, \) we have
\[
(a - b)^2 \geq 0 \\
a^2 - 2ab + b^2 \geq 0.
\]
Then
\[
(3) \quad a^2 + b^2 \geq 2ab.
\]

2. **Main Results**

In this section, we have a trace equality related with Hadamard product and Kronecker product. Also, we obtain some inequalities for traces of products of matrices.

The following theorem give the relation between trace of Hadamard product and trace of Kronecker product.

**Theorem 1.** Let \( A = (a_{ij}) \) and \( B = (b_{ij}) \) be \( n \times n \) matrices. Then
\[
(4) \quad tr(A \otimes B) = n.tr(A \circ B) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (a_{ii} - a_{jj})(b_{ii} - b_{jj}).
\]
Proof. By Lemma 2, we have

\[
tr(A)tr(B) = n.tr(A \circ B) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (a_{ii} - a_{jj})(b_{ii} - b_{jj}).
\]

Let us use the principle of mathematical induction on \( n \).
For \( n = 2 \),

\[
\sum_{i=1}^{2} a_{ii} \sum_{i=1}^{2} b_{ii} = 2 \sum_{i=1}^{2} a_{ii}b_{ii} - 2 \sum_{i=1}^{2} (a_{ii} - a_{jj})(b_{ii} - b_{jj}),
\]

where

\[
tr(A) = \sum_{i=1}^{2} a_{ii}, \quad tr(B) = \sum_{i=1}^{2} b_{ii}, \quad tr(A \circ B) = \sum_{i=1}^{2} a_{ii}b_{ii}.
\]

Let us rewrite the right hand side of above equation as follows;

\[
\sum_{i=1}^{2} a_{ii} \sum_{i=1}^{2} b_{ii} = 2 \sum_{i=1}^{2} a_{ii}b_{ii} - \sum_{i=1}^{2} (a_{ii} - a_{jj})(b_{ii} - b_{jj}) - \sum_{i=1}^{2} a_{ii}b_{ii}.
\]

Then it is clear that the result is hold for \( n = 2 \).

Assume that it is true for all positive integers \( n = k \). That is,

\[
\sum_{i=1}^{k} a_{ii} \sum_{i=1}^{k} b_{ii} = k \sum_{i=1}^{k} a_{ii}b_{ii} - \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} (a_{ii} - a_{jj})(b_{ii} - b_{jj}).
\]

We have to show that it is true for \( n = k + 1 \). That is,

\[
\sum_{i=1}^{k+1} a_{ii} \sum_{i=1}^{k+1} b_{ii} = (k + 1) \sum_{i=1}^{k+1} a_{ii}b_{ii} - \sum_{i=1}^{k} \sum_{j=i+1}^{k+1} (a_{ii} - a_{jj})(b_{ii} - b_{jj}).
\]

If we add

\[k.a_{(k+1)(k+1)}b_{(k+1)(k+1)} + \sum_{i=1}^{k+1} a_{ii}b_{ii} - \sum_{i=1}^{k} (a_{ii} - a_{(k+1)(k+1)})(b_{ii} - b_{(k+1)(k+1)})\]

term to the right and the left sides of equation (6), then we obtain equation (7). Therefore, the result is true for every \( n \geq 2 \).
In the following theorem, for arbitrary square matrices $A$ and $B$, we obtain the trace inequality for Hadamard product of matrix sums.

**Theorem 2.** Let $A = (a_{ij})$ and $B = (b_{ij})$ be $n \times n$ matrices. Then

$$
tr(A \circ B) \leq tr \left( \left( \frac{A + B}{2} \right) \circ \left( \frac{A + B}{2} \right) \right).
$$

**Proof.** From $tr(A \circ B) = \sum_{i=1}^{n} a_{ii}b_{ii}$, we have

$$
tr \left( \left( \frac{A + B}{2} \right) \circ \left( \frac{A + B}{2} \right) \right) = \sum_{i=1}^{n} \left( \frac{a_{ii} + b_{ii}}{2} \right) \circ \left( \frac{a_{ii} + b_{ii}}{2} \right)
$$

$$
= \frac{1}{4} \sum_{i=1}^{n} (a_{ii} + b_{ii})(a_{ii} + b_{ii})
$$

$$
= \frac{1}{4} \sum_{i=1}^{n} (a_{ii}^2 + 2a_{ii}b_{ii} + b_{ii}^2)
$$

$$
= \sum_{i=1}^{n} \frac{a_{ii}b_{ii}}{2} + \frac{1}{4} \sum_{i=1}^{n} (a_{ii}^2 + b_{ii}^2).
$$

Then from equation (3) we obtain

$$
tr \left( \left( \frac{A + B}{2} \right) \circ \left( \frac{A + B}{2} \right) \right) = \sum_{i=1}^{n} \frac{a_{ii}b_{ii}}{2} + \frac{1}{4} \sum_{i=1}^{n} (a_{ii}^2 + b_{ii}^2)
$$

$$
\geq \sum_{i=1}^{n} \frac{a_{ii}b_{ii}}{2} + \frac{1}{4} \sum_{i=1}^{n} 2a_{ii}b_{ii}
$$

$$
= \sum_{i=1}^{n} \frac{a_{ii}b_{ii}}{2} + \sum_{i=1}^{n} \frac{a_{ii}b_{ii}}{2}
$$

$$
= \sum_{i=1}^{n} a_{ii}b_{ii} = tr(A \circ B).
$$

By considering above theorem, we give a generalization for trace inequality of Hadamard product.

**Theorem 3.** Let $A^{(j)}$ be $n \times n$ matrices having positive real numbers as diagonal elements for every $j \in \mathbb{N}$. Then

$$
tr \left( \bigcirc_{j=1}^{m} A^{(j)} \right) \leq tr \left( \bigcirc_{j=1}^{m} \frac{A^{(j)}}{m} \right).
$$
Proof. We write

\[
\text{tr} \left( \bigcirc_{j=1}^{m} A^{(j)} \right) = \sum_{i=1}^{n} a_{ii}^{(1)} a_{ii}^{(2)} \cdots a_{ii}^{(m)}
\]

\[
= \sum_{i=1}^{n} \prod_{j=1}^{m} a_{ii}^{(j)} .
\]

Also we can write

\[
\text{tr} \left( \bigcirc \sum_{j=1}^{m} \frac{A^{(j)}}{m} \right) = \text{tr} \left( \bigcirc \left( \frac{A^{(1)} + A^{(2)} + \cdots + A^{(m)}}{m} \right) \right)
\]

\[
= \text{tr} \left[ \left( \frac{A^{(1)} + A^{(2)} + \cdots + A^{(m)}}{m} \right) \circ \cdots \circ \left( \frac{A^{(1)} + A^{(2)} + \cdots + A^{(m)}}{m} \right) \right]
\]

\[
= \sum_{i=1}^{n} \left( \frac{a_{ii}^{(1)} + a_{ii}^{(2)} + \cdots + a_{ii}^{(m)}}{m} \right)^{m} .
\]

Thus from Lemma 1 and equation (8) we have

\[
\text{tr} \left( \bigcirc \sum_{j=1}^{m} \frac{A^{(j)}}{m} \right) = \sum_{i=1}^{n} \left( \frac{a_{ii}^{(1)} + a_{ii}^{(2)} + \cdots + a_{ii}^{(m)}}{m} \right)^{m}
\]

\[
\geq \prod_{i=1}^{n} \prod_{j=1}^{m} a_{ii}^{(j)} = \text{tr} \left( \bigcirc_{j=1}^{m} A^{(j)} \right) .
\]

Example 1. Let \( A = \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix} \) and \( B = \begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix} \) be matrices. Then we compute \( A \circ B, A \otimes B, \text{tr}(A \circ B), \text{tr}(A \otimes B) \) as follows;

\[
A \circ B = \begin{pmatrix} 12 & -1 \\ 12 & -2 \end{pmatrix}, \quad A \otimes B = \begin{pmatrix} 12 & -3 & 4 & -1 \\ 9 & 6 & 3 & 2 \\ 16 & -4 & -4 & 1 \\ 12 & 8 & -3 & -2 \end{pmatrix},
\]

\[
\text{tr}(A \circ B) = 10, \quad \text{tr}(A \otimes B) = 12
\]

By applying Theorem 1, we have

\[
2 \text{tr}(A \circ B) - \sum_{i=1}^{1} \sum_{j=i+1}^{2} (a_{ii} - a_{jj})(b_{ii} - b_{jj}) = 2.10 - (a_{11} - a_{22})(b_{11} - b_{22})
\]

\[
= 20 - 8
\]

\[
= 12
\]

\[
= \text{tr}(A \otimes B)
\]
Therefore, the theorem is true.

**Example 2.** Let \( A = \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix} \) and \( B = \begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix} \) be matrices. Then

\[
\text{tr}(A \circ B) = 10, \quad \text{tr} \left( \frac{A + B}{2} \circ \frac{A + B}{2} \right) = \frac{53}{4}.
\]

By applying Theorem 2, we have

\[
10 \leq \frac{53}{4}.
\]

Therefore, the theorem is true.

**References**