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Self Similar Surfaces in Euclidean Spaces

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Abstract. In the present study we consider self-similar surfaces imbedded in Euclidean space. We give necessary and sufficient conditions for the surface of revolution and surfaces with Monge patch in $\mathbb{E}^3$ to become self-similar. Further, we investigate self-similar surfaces in Euclidean 4-space $\mathbb{E}^4$.Additionally we give necessary and sufficient condition of spherical product surfaces and smooth surfaces given with the Monge patch in $\mathbb{E}^4$ to become self-similar.

Key words: Monge patch; Surface of revolution surface; Self-similar surface.
AMS Classification: 53C40, 53C42.

1. Introduction

The mean curvature flow (MCF) is the gradient flow of the area functional on the space of n-dimensional submanifolds of a given Riemannian manifold. From the viewpoint of analysis, this flow is governed by a non-linear parabolic equation. Self-similar flows arise as special solutions of the MCF that preserve the shape of the evolving submanifold. The simplest and most important example of a self-similar flow is when the evolution is a homothety. Such a self-similar submanifold M with mean curvature vector H satisfies the following non-linear, elliptic system:

$$H + \lambda X^{\perp} = 0$$

where $X^{\perp}$ stands for the projection of the position vector X onto the normal space. If λ is any strictly positive constant, the submanifold shrinks in finite time to a single point under the action of the MCF, its shape remaining unchanged. If λ is strictly negative, the submanifold will expand, its shape compact. The case of vanishing λ is the well-known case of a minimal submanifold, which of course is stationary under the action of the flow.
Before stating our own results, we mention some work that has been done on the subject: the planar curves which are self-shrinking where classified in [2];

In particular, the only simple self-shrinking curves are the round circles. In [7], the existence of non-spherical self-similar hypersurfaces of revolution in $\mathbb{E}^n$ were shown; in [4], the author described rotational symmetric Lagrangian self-shrinkers and self-expanders in $\mathbb{E}^{2n}$. Very recently, wider classes of self-similar Lagrangian submanifolds has been derived in [14]. Further, it was shown in [3] and [6] that the only Lagrangian self-similar submanifolds of $\mathbb{E}^{2n}$ which are foliated by (n-1)-dimensional spheres are the examples found in [4] ; in another direction spherical self-shrinkers have been characterized in [16]. In [5] the author give a characterization of the only self-similar surfaces of $\mathbb{E}^3$ known until now: he first prove that the self-similar surfaces of revolution discovered by Angenent [7] are the only cyclic self-similar surfaces, and next that the cylinders over planar self-similar curves are the only ruled self-similar surfaces.
The paper is organized as follows: Section 2 provides some basic concepts of the surfaces in $\mathbb{E}^n$. In Section 3 we give some results for the self-similar surfaces in $\mathbb{E}^3$. In Section 4 we investigate self-similar surfaces in Euclidean 4-space $\mathbb{E}^4$. Additionally we give necessary and sufficient condition of spherical product surfaces and surfaces with Monge patch in $\mathbb{E}^4$ to become self-similar.

2. Basic Concepts

Let M be a smooth surface in $\mathbb{E}^n$ given with the patch $X(u,v) : (u,v) \in D \subset \mathbb{E}^2$. The tangent space to M at an arbitrary point $p = X(u,v)$ of M span $\{X_u, X_v\}$. In the chart (u,v) the coefficients of the first fundamental form of M are given by

$$E = \langle X_u, X_u \rangle, F = \langle X_u, X_v \rangle, G = \langle X_v, X_v \rangle, \tag{1}$$

where $\langle , \rangle$ is the Euclidean inner product. We assume that $g = EG - F^2 \neq 0$, i.e. the surface patch $X(u,v)$ is *regular*.
For each $p \in M$, consider the decomposition $T_p\mathbb{E}^n = T_pM \oplus T_p^{\perp}M$ where $T_p^{\perp}M$ is the orthogonal component of T_pM in $\mathbb{E}^n$. Let $\widetilde{\nabla}$ be the Riemannian connection of $\mathbb{E}^4$. Given any local vector fields X_1, X_2 tangent to M. The induced Riemannian connection on M is defined by

$$\nabla_{X_1}X_2 = (\widetilde{\nabla}_{X_1}X_2)^T,$$

where T meaning the tangent component.
Let $\chi(M)$ and $\chi^{\perp}(M)$ be the space of the smooth vector fields tangent to M and the space of the smooth vector fields normal to M, respectively. Consider the *second fundamental map*

$$\begin{aligned} &h : \chi(M) \times \chi(M) \to \chi^{\perp}(M) \\ &h(X_i, X_j) = \widetilde{\nabla}_{X_i}X_j - \nabla_{X_i}X_j \quad 1 \leq i,j \leq 2. \end{aligned} \tag{2}$$

This map is well-defined, symmetric and bilinear.

For any arbitrary orthonormal frame field $\{N_1, N_2, ..., N_{n-2}\}$ of M, recall the shape operator

$$\begin{aligned} A &: \chi^{\perp}(M) \times \chi(M) \to \chi(M) \\ A_{N_i} X &= -(\widetilde{\nabla}_{X_i} N_i)^T, \quad X_i \in \chi(M). \end{aligned}$$

This operator is bilinear, self-adjoint and for any tangent vector fields X_1, $X_2 \in T_pM$ satisfies the following equation:

$$\langle A_{N_k} X_j, X_i \rangle = \langle h(X_i, X_j), N_k \rangle = c_{ij}^k, \tag{3}$$

where $1 \leq i, j \leq 2, 1 \leq k \leq n-2$.
The equation (2) is called Gaussian formula, where

$$\nabla_{X_i} X_j = \sum_{k=1}^{2} \Gamma_{ij}^k X_k \ , \quad 1 \leq i, j \leq 2$$

and

$$h(X_i, X_j) = \sum_{k=1}^{n-2} c_{ij}^k N_k \ , \qquad 1 \leq i, j \leq 2. \tag{4}$$

Here Γ_{ij}^k are called *Christoffel symbols* and c_{ij}^k are the *coefficients of the second fundamental form* [13].
Further, the mean curvature vector of a regular patch $X(u, v)$ is defined by

$$\overrightarrow{H} = \frac{1}{2W^2} \sum_{k=1}^{n-2} (c_{11}^k G + c_{22}^k E - 2c_{12}^k F) N_k. \tag{5}$$

Recall that a surface in $\mathbb{E}^n$ is said to be *minimal* if its mean curvature vanishes identically [10].

Definition 1. *A surface is self-similar if the mean curvature vector H satisfies the following non-linear, elliptic system:*

$$\overrightarrow{H} + \lambda X^{\perp} = 0$$

where $X^{\perp}$ stands for the projection of the position vector X onto the normal space [6]

3. Self-Similar Surfaces in Euclidean 3-Spaces

Let $X : U \to \mathbb{E}^3$ a local parametrization of a surface M . We denote by the coefficients of the first fundamental form E, F and G of the surface M are given by (1). Let N be the unit normal vector field of M then the coefficients of the second fundamental form are defined to be:

$$e = \langle X_{uu}, N\rangle, f = \langle X_{uv}, N\rangle, \ g = \langle X_{vv}, N\rangle.$$

In order to simplify further calculations, we introduce the following coefficients, which are proportional to the previous ones:

$$\widetilde{e} = \langle X_{uu}, X_u \times X_v\rangle, \widetilde{f} = \langle X_{uv}, X_u \times X_v\rangle, \ \widetilde{g} = \langle X_{vv}, X_u \times X_v\rangle. \tag{6}$$

Rather than the classical formula for the mean curvature,

$$2H = \frac{eG + gE - 2fF}{EG - F^2},$$

it will be more convenient to use the following one (see, [5]) :

$$2H = \frac{\widetilde{e}G + \widetilde{g}E - 2\widetilde{f}F}{(EG - F^2)^{3/2}}. \tag{7}$$

In codimension one, the self-similar equation $\overrightarrow{H} + \lambda X^{\perp} = 0$ becomes scalar, namely: $H + \lambda < X, N >= 0$. Moreover, in $\mathbb{E}^3$ we have:

$$\langle X, N\rangle = \frac{1}{EG - F^2}\langle X, X_u \times X_v\rangle = \frac{1}{EG - F^2}\det(X, X_u, X_v). \tag{8}$$

Finally, from Equations (7) and (8) we deduce:

Lemma 1. [5] A surface M of $\mathbb{E}^3$ is self-similar if and only if, for any local parametrization $X : U \to \mathbb{E}^3$ of M, the following formula holds:

$$\widetilde{e}G + \widetilde{g}E - 2\widetilde{f}F + 2\lambda(EG - F^2)det(X, X_u, X_v) = 0. \tag{9}$$

Corollary 1. *Let M be a smooth surface of $\mathbb{E}^3$ with constant mean curvature. Then M is self-similar if and only if $(EG - F^2)det(X, X_u, X_v)$ is a nonzero constant.*

Remark 1. *The equality $\widetilde{e}G + \widetilde{g}E - 2\widetilde{f}F = 0$ implies that M is minimal. So, the case of vanishing λ is the well-known case of a minimal surface.*

We give the following results.

Proposition 1. *Let M be a surface of revolution given with the regular patch*

$$X(u, v) = (f(u), g(u)\cos v, g(u)\sin v)). \tag{10}$$

Then M is a self similar surface if and only if the following equality holds:

$$g(g_u f_{uu} - f_u g_{uu}) + f_u\left(f_u^2 + g_u^2\right) + 2\lambda g\left(f_u^2 + g_u^2\right)(f g_u - g f_u) = 0. \tag{11}$$

Proof. Differentiating (10) and using (6) and (9) we get the result. ■

As a consequence of Proposition 1 we obtain the following result.

Corollary 2. *Let M be a surface of revolution given with the regular patch (10). Then M is a self similar surface if and only if the following formulas hold;*

$$\begin{aligned} g\left(g_u f_{uu} - f_u g_{uu}\right) + f_u\left(f_u^2 + g_u^2\right) &= c_1, \\ g\left(f_u^2 + g_u^2\right)\left(f g_u - g f_u\right) &= c_2, \end{aligned}$$

where c_1 and c_2 are nonzero constant.

Example 1. *The surface of revolution given with the parametrization $f(u) = u$ and $g(u) = a\cos h(\frac{u}{a} + b)$ is a minimal surface (i.e. catenoid) and for $\lambda = 0$ it is self-similar.*

Example 2. *The surface of revolution given with the parametrization $f(u) = u$ and $g(u) = const.$ is a flat surface (i.e. cylinder) and for $g = \pm\frac{1}{\sqrt{2\lambda}}$ it is self-similar.*

Proposition 2. *Let M be smooth surface given with the Monge patch*

$$X(u,v) = (u, v, f(u,v)). \tag{12}$$

Then M is a self similar surface if and only if

$$f_{uu}\left(1 + f_v^2\right) + f_{vv}\left(1 + f_u^2\right) - 2f_u f_v f_{uv} + 2\lambda\left(1 + f_u^2 + f_v^2\right)\left(f - u f_u - v f_v\right) = 0$$

holds.

Proof. Differentiating (12) and using (6) and (9) we get the result. ■

Corollary 3. *Let M be a smooth surface given with the Monge patch (12). Then M is a self similar surface if and only if the following formulas hold;*

$$\begin{aligned} f_{uu}\left(1 + f_v^2\right) + f_{vv}\left(1 + f_u^2\right) - 2f_u f_v f_{uv} &= c_1, \\ \left(1 + f_u^2 + f_v^2\right)\left(f - u f_u - v f_v\right) &= c_2, \end{aligned} \tag{13}$$

where c_1 and c_2 are nonzero constants.

Definition 2. *The surface M defined as the sum of two space curves $\alpha(u) = (u, 0, f(u))$ and $\beta(v) = (0, v, g(v))$ is called a translation surface in E^3. So, the translation surfaces are defined with the Monge patch*

$$X(u,v) = (u, v, f(u) + g(v)). \tag{14}$$

The following results are well-known;

Example 3. *The surface of revolution given with the parametrization* $f(u) = \frac{1}{a}\log|\cos(au)|$ *and* $g(v) = -\frac{1}{a}\log|\cos(av)|$ *(a is a nonzero constant) is a minimal surface (i.e. surface of Scherk) and for* $\lambda = 0$ *it is self-similar. For more detail see [15].*

Theorem 1. [15] *Let* M *be a translation surface with constant mean curvature* $H \neq 0$ *in 3-dimensional Euclidean space* E^3*. Then* M *is congruent to a surface given with the parametrization*

$$\begin{aligned} f(u) &= \frac{-\sqrt{1-a^2}}{2H}\sqrt{1-4H^2u^2} \\ g(v) &= -av. \end{aligned} \tag{15}$$

where $a < 1$ *is non-zero positive constant.*

By the use of (13)-(15) we get the following result.

Corollary 4. *Translation surface with constant mean curvature* $H \neq 0$ *in 3-dimensional Euclidean space* $\mathbb{E}^3$ *can not be self-similar.*

4. Self-Similar Surfaces in Euclidean 4-Spaces

2-dimensional surfaces in $\mathbb{E}^4$ are interesting object for investigation of geometers. Here we have some difficult problems which wait its solutions. Hence the investigation of various classes of surfaces in $\mathbb{E}^4$ with point of view of influence of the principal invariant -the vector of mean curvature H on the behavior of surfaces is an actual problem.

Definition 3. *Let* $X : U \to \mathbb{E}^4$ *a local parametrization of a surface* M*. For a self similar surface* M *in* $\mathbb{E}^4$ *the following equality holds:*

$$\langle H, N_i \rangle + \lambda \langle X, N_i \rangle = 0; i = 1, 2.$$

That is equivalent to

$$\langle X_{uu}, N_i \rangle G + \langle X_{vv}, N_i \rangle E - 2\langle X_{uv}, N_i \rangle F = -2\lambda(EG - F^2)\langle X, N_i \rangle . \tag{16}$$

We deduce that the surface given with a regular patch $X(u, v)$ in $\mathbb{E}^4$ is self similar if and only if the following equalities hold:

$$\begin{aligned} c_{11}^1 G + c_{22}^1 E - 2c_{12}^1 F &= -2\lambda(EG - F^2)\langle X, N_1 \rangle , \\ c_{11}^2 G + c_{22}^2 E - 2c_{12}^2 F &= -2\lambda(EG - F^2)\langle X, N_2 \rangle \end{aligned} \tag{17}$$

Definition 4. *Let $\alpha : \mathbb{R} \longrightarrow \mathbb{E}^3$ be an Euclidean space curve and $\beta : \mathbb{R} \longrightarrow \mathbb{E}^2$ Euclidean plane curve. Put $\alpha(u) = (f_1(u), f_2(u), f_3(u))$ and $\beta(v) = (g_1(v), g_2(v))$. Then their spherical product patch is given by*

$$X = \alpha \otimes \beta : \mathbb{E}^2 \longrightarrow \mathbb{E}^4;\ X(u,v) = (f_1(u), f_2(u), f_3(u)g_1(v), f_3(u)g_2(v));$$

$u \in I = (u_0, u_1), v \in J = (v_0, v_1)$, which is a surface in $\mathbb{E}^4$.

For the case $f_1(u) = 0$ or $f_2(u) = 0$, the patch $X = \alpha \otimes \beta : \mathbb{E}^2 \longrightarrow \mathbb{E}^3$ becomes a spherical product of two $2D$ curves. Recently, G. Ganchev and V. Milousheva considered the general product of the space curve $\alpha(u) = (f_1(u), f_2(u), f_3(u))$ with the circle $\beta(v) = (\cos v, \sin v)$ such that

$$X(u,v) = \alpha(u) \otimes \beta(v) = (f_1(u), f_2(u), f_3(u)\cos v, f_3(u)\sin v); \tag{18}$$

$u \in I$, $0 \leq v < 2\pi$, where $\alpha(u)$ is parametrized with respect to the arc-length, i.e. $({f_1}')^2 + ({f_2}')^2 + ({f_3}')^2 = 1$ and $f_3(u) > 0$, [12].
We obtain the following result.

Proposition 3. *Let M be a spherical product surface given with the regular patch (18). Then M is a self similar surface if and only if*

$$\kappa^2 f_3 - {f_3}'' + 2\lambda f_3 \left(f_1{f_1}'' + f_2{f_2}'' + f_3{f_3}''\right) = 0$$

and

$$2\lambda f_3 \left\{f_1\left({f_2}'{f_3}'' - {f_3}'{f_2}''\right) + f_2\left({f_1}'{f_3}'' - {f_3}'{f_1}''\right) + f_3\left({f_1}'{f_2}'' - {f_2}'{f_1}''\right)\right\} - \kappa_1 = 0$$

hold. Where κ is the Frenet curvature of the curve α and $\kappa_1 = {f_1}'{f_2}''(u) - {f_1}''{f_2}'(u)$ is the curvature of the projection of α on the Oe_1e_2- plane.

Proof. The tangent space of M is spanned by the vector fields

$$\begin{aligned}\frac{\partial X}{\partial u} &= (f_1'(u), f_2'(u), f_3'(u)\cos v, f_3'(u)\sin v),\\ \frac{\partial X}{\partial v} &= (0, 0, -f_3(u)\sin v, f_3(u)\cos v).\end{aligned}$$

Hence, the coefficients of the first fundamental form of the surface are

$$\begin{aligned}E &= \langle X_u(u,v), X_u(u,v)\rangle = 1,\\ F &= \langle X_u(u,v), X_v(u,v)\rangle = 0,\\ G &= \langle X_v(u,v), X_v(u,v)\rangle = f_3(u)^2,\end{aligned}$$

where $\langle , \rangle$ is the standard scalar product in $\mathbb{R}^4$.The second partial derivatives of $X(u,v)$ are expressed as follows

$$\begin{aligned}X_{uu}(u,v) &= ({f_1}''(u), {f_2}''(u), {f_3}''(u)\cos v, {f_3}''(u)\sin v),\\ X_{uv}(u,v) &= (0, 0, -f_3'(u)\sin v, f_3'(u)\cos v),\\ X_{vv}(u,v) &= (0, 0, -f_3(u)\cos v, -f_3(u)\sin v).\end{aligned} \tag{19}$$

Further, the normal space of M is spanned by the vector fields

$$
\begin{aligned}
N_1 &= \frac{1}{\kappa}(f_1{}'', f_2{}'', f_3{}'' \cos v, f_3{}'' \sin v) \\
N_2 &= \frac{1}{\kappa}(f_2{}' f_3{}'' - f_3{}' f_2{}'', f_3{}' f_1{}'' - f_1{}' f_3{}'', \\
&\quad (f_1{}' f_2{}'' - f_2{}' f_1{}'') \cos v, (f_1{}' f_2{}'' - f_2{}' f_1{}'') \sin v).
\end{aligned} \tag{20}
$$

Using (3), (19) and (20) we can calculate the coefficients of the second fundamental form c_{ij}^k are as follows *[8]*:

$$
\begin{aligned}
c_{11}^1 &=< X_{uu}(u,v), N_1 >= \kappa, \\
c_{12}^1 &=< X_{uv}(u,v), N_1 >= 0, \\
c_{22}^1 &=< X_{vv}(u,v), N_1 >= -\frac{f_3 f_3{}''}{\kappa}, \\
c_{11}^2 &=< X_{uu}(u,v), N_2 >= 0, \\
c_{12}^2 &=< X_{uv}(u,v), N_2 >= 0, \\
c_{22}^2 &=< X_{vv}(u,v), N_2 >= \frac{-\kappa_1 f_3}{\kappa}.
\end{aligned} \tag{21}
$$

Substituting (21) in to (17), after some calculation we get the result. ∎

Corollary 5. *Let M be a spherical product surface given with the regular patch (18). If $\kappa^2 f_3 - f_3{}'' = 0$ and $\kappa_1 = 0$ then M is a minimal surface and for $\lambda = 0$ it is self-similar.*

For more detail see [9].

Definition 5. *In the considering work we use the representation of surfaces in the explicit form*

$$
X(u,v) = (u, v, f(u,v), g(u,v)), \tag{22}
$$

where f and g are some smooth functions. The parametrization (22) is called Monge patch in E^4 (see, [1]).

First we obtain the following result.

Proposition 4. *Let M be a smooth surface given with the Monge patch (22). Then M is a self similar surface if and only if*

$$
f_{uu}G + f_{vv}E - 2f_{uv}F + 2\lambda\left(EG - F^2\right)(f - uf_u - vf_v) = 0
$$

$$
\begin{aligned}
&(Ag_{uu} - Bf_{uu})\,G + (Ag_{vv} - Bf_{vv})\,E - 2\,(Ag_{uv} - Bf_{uv})\,F \\
&+2\lambda\left(EG - F^2\right)\{Ag - Bf + u\,(Bf_u - Ag_u) + v\,(Bf_v - Ag_v)\} = 0
\end{aligned}
$$

hold, where

$$\begin{aligned} E &= 1+(f_u)^2+(g_u)^2, \\ F &= f_u f_v + g_u g_v, \\ G &= 1+(f_v)^2+(g_v)^2, \end{aligned}$$

and

$$\begin{aligned} A &= 1+(f_u)^2+(f_v)^2, \\ B &= f_u g_u + f_v g_v\ , \\ C &= 1+(g_u)^2+(g_v)^2. \end{aligned}$$

Proof. The tangent space of M is spanned by the vector fields

$$\begin{aligned} \frac{\partial X}{\partial u} &= (1,0,f_u,g_u), \\ \frac{\partial X}{\partial v} &= (0,1,f_v,g_v). \end{aligned}$$

Hence, the coefficients of the first fundamental form of the surface are

$$\begin{aligned} E = &< X_u(u,v), X_u(u,v) > &= 1+(f_u)^2+(g_u)^2, \\ F = &< X_u(u,v), X_v(u,v) > &= f_u f_v + g_u g_v, \\ G = &< X_v(u,v), X_v(u,v) > &= 1+(f_v)^2+(g_v)^2, \end{aligned} \tag{23}$$

where $\langle , \rangle$ is the standard scalar product in $\mathbb{E}^4$.The second partial derivatives of $X(u,v)$ are expressed as follows

$$\begin{aligned} X_{uu}(u,v) &= (0,0,f_{uu},g_{uu}), \\ X_{uv}(u,v) &= (0,0,f_{uv},g_{uv}), \\ X_{vv}(u,v) &= (0,0,f_{vv},g_{vv}). \end{aligned} \tag{24}$$

Further, the normal space of M is spanned by the vector fields

$$\begin{aligned} N_1 &= \frac{1}{\sqrt{A}}(-f_u,-f_v,1,0) \\ N_2 &= \frac{1}{W\sqrt{A}}(Bf_u - Ag_u, Bf_v - Ag_v, -B, A). \end{aligned} \tag{25}$$

Using (5), (24) and (22) we can calculate the coefficients of the second fundamental form h are as follows *[9]*:

$$
(26)\qquad
\begin{aligned}
c_{11}^{1} &=< X_{uu}(u,v), N_1 >= \frac{f_{uu}}{\sqrt{A}},\\
c_{12}^{1} &=< X_{uv}(u,v), N_1 >= \frac{f_{uv}}{\sqrt{A}},\\
c_{22}^{1} &=< X_{vv}(u,v), N_1 >= \frac{f_{vv}}{\sqrt{A}},\\
c_{11}^{2} &=< X_{uu}(u,v), N_2 >= \frac{-Bf_{uu}+Ag_{uu}}{W\sqrt{A}},\\
c_{12}^{2} &=< X_{uv}(u,v), N_2 >= \frac{-Bf_{uv}+Ag_{uv}}{W\sqrt{A}},\\
c_{22}^{2} &=< X_{vv}(u,v), N_2 >= \frac{-Bf_{vv}+Ag_{vv}}{W\sqrt{A}}.
\end{aligned}
$$

So, substituting (26) into (17) we get the result. ■

Definition 6. *The surface given with the parametrization (22) by the parametrization*

$$
(27)\qquad f(u,v) = f_3(u) + g_3(v),\ \ g\ (u,v) = f_4(u) + g_4(v)
$$

is called translation surface in Euclidean 4-space $\mathbb{E}^4$ [11].

The following results are well-known;

Corollary 6. *Let* M *be a translation surface of* $\mathbb{E}^4$ *given with (27). If* M *is given with the parametrization*

$$
\begin{aligned}
f_k(u) &= \frac{c_k}{c_3^2+c_4^2}\left(\log\left|\cos(\sqrt{a}u)\right| + cu\right) + e_k u,\\
g_k(v) &= \frac{c_k}{c_3^2+c_4^2}\left(-\log\left|\cos(\sqrt{b}v)\right| + dv\right) + p_k v,\ \ k=3,4,
\end{aligned}
$$

then M *is a minimal surface and for* $\lambda = 0$ *it is is self-similar.*

For more detail see [11].

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