Selçuk J. Appl. Math. Vol. 11. No.1. pp. 81-94 , 2010 Selçuk Journal of Applied Mathematics

Age and Block Replacement Policies in Renewal Processes

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Received Date: March 24, 2009 Accepted Date: September 4, 2009

Abstract. In this study, renewal processes, age and block replacement policies are described. Certain classes of distribution functions which are related to the usage of these policies are given. Some results on age and block replacement policies in renewal processes are reviewed.

Key words: Age replacement policy; block replacement policy; renewal theory; stochastic order; Laplace order. 2000 Mathematics Subject Classification: 60K05.

1. Introduction

In many situations, failure of a unit during actual operation is costly or dangerous. Replacement policies are followed to reduce the incidence of system failure. Some useful replacement policies in popular use are the age replacement policy and the block replacement policy. Under an age replacement policy, a unit is replaced upon failure or at age T, a specified positive constant, whichever comes first. Under block replacement policy, a unit is replaced upon a failure and at times T, 2T, 3T,...

Age replacement is administratively more difficult to implement, since the age of the unit must be recorded. Block replacement is simpler to administer since the age of the unit need not to be recorded. It leads to more frequent replacement of relatively new items. This type of policy is commonly used with computers and other complex electronic systems. On the other hand, age replacement is more flexible since planned replacement takes into account the age of the unit under this policy. Therefore, it is some of interest to compare these two policies with respect to the number of failures, the number of planned replacements, and the number of removals. Age and block replacement policies have been investigated by Barlow and Proschan (1975) and Yue and Cao (2001), among others. All the replacements can be treated by the techniques of renewal theory. Before investigating the replacement problem, it is necessary to present some classes of life distributions and the renewal theory.

2. Some Classes of Life Distributions

Assume that the life length of a unit has a distribution function F with F(x) = 0for x < 0. The corresponding survival function is denoted by $\overline{F} = 1 - F$.

Increasing (Decreasing) Failure Rate

A distribution function F is IFR (DFR) if for all x > 0, $\overline{F}(x+t)/\overline{F}(t)$ is decreasing (increasing) in t whenever $t \ge 0$ and $\overline{F}(t) > 0$. If F has a density function f, then this is equivalent to the condition that the failure rate $r(t) = f(t)/\overline{F}(t)$ is increasing (decreasing) in t on $\{t : \overline{F}(t) > 0\}$.

New Better (Worse) Than Used

A distribution function F is NBU (NWU) if

(1)
$$\overline{F}(x+y) \le (\ge)\overline{F}(x) \overline{F}(y)$$

for $x \ge 0$, $y \ge 0$. Equality in (1) holds if and only if F is the exponential distribution function.

New Better (Worse) Than Used in Expectation

A distribution function F is NBUE (NWUE) if

i) F has finite (or infinite mean) μ ,

ii) $\int_t^{\infty} \overline{F}(x) dx \leq (\geq) \mu \overline{F}(t)$ for $t \geq 0$ where $\int_t^{\infty} [\overline{F}(x)/\overline{F}(t)] dx$ represents the conditional mean remaining life of a unit of age t.

It is well known that

$$IFR \Rightarrow NBU \Rightarrow NBUE,$$

and

$$DFR \Rightarrow NWU \Rightarrow NWUE$$

New Better (Worse) Than Used in Laplace Ordering

Let X and Y be two nonnegative random variables with distribution functions F and G, respectively. It is said that X is smaller than Y (or F is smaller than G) with respect to Laplace order \leq_L , denoted by $X \leq_L Y (F \leq_L G)$ if

$$\int_0^\infty e^{-st} dF(t) \ge \int_0^\infty e^{-st} dG(t), \qquad s \ge 0.$$

Let X_t denote the residual life of X at age $t \ge 0$. The nonnegative random variable X is said to be NBUL(NWUL) if and only if $X_t \le_L (\ge_L) X$ or equivalently

$$\int_0^\infty e^{-sx}\overline{F}(t+x)dx \le (\ge)\overline{F}(t)\int_0^\infty e^{-sx}\overline{F}(x)dx.$$

Wang (1996) has shown that the following expressions hold;

$$NBU \Rightarrow NBUL \Rightarrow NBUE, \qquad NWU \Rightarrow NWUL \Rightarrow NWUE.$$

3. Renewal Theory

A renewal process is a sequence of independent, identically distributed, nonnegative random variables $X_1, X_2, ...$, which, with probability 1, are not all zero. Let F be the distribution function of X_1 ; F is called the *underlying distribution* function of the renewal process. F^{k*} , the k-fold Stieltjes convolution of F with itself, is the distribution function of $S_k \equiv X_1 + X_2 + ... + X_k$.

Renewal theory is primarily concerned with the number of renewals N(t) in [0,t]. N(t), the renewal random variable is the maximum value of k for which $S_k \leq t$. The stochastic process $\{N(t), t \geq 0\}$ is also known as a renewal counting process (Ross 1983).

It is known that $P(N(t)\geq n)=P(S_n\leq t)=F^{n*}(t)$ for $n=0,1,\ldots$. It follows that $P(N(t)=n)=P(N(t)\geq n)-P(N(t)\geq n+1),$ so that

$$P(N(t) = n) = F^{n*}(t) - F^{(n+1)*}(t), \qquad n = 0, 1, \dots$$

The mean value function (renewal function) of the renewal process $\{N(t), t \ge 0\}$ is;

$$\begin{split} M(t) &= E(N(t)) \\ &= \sum_{k=1}^{\infty} P(N(t) \geq k) \\ &= \sum_{k=1}^{\infty} F^{k*}(t) , \qquad t \geq 0 \end{split}$$

It is well known that the renewal function M(t) satisfies the integral equation (renewal equation);

(2)
$$M(t) = F(t) + \int_0^t M(t-x)dF(x), \quad t \ge 0.$$

If F has a density f, differentiation of (2) yields

$$m(t) = f(t) + \int_0^t m(t-x)f(x)dx, \qquad t \ge 0,$$

where $m(t) = \frac{d}{dt}M(t)$ is known as the renewal density.

Note that N(t) is the number of renewals in [0, t], and the next arrival will be that numbered N(t) + 1. That is to say, we have begun our observation at a point in the random interval $I_t = [S_{N(t)}, S_{N(t)+1})$, the endpoints of which are arrival times. $E(t) = S_{N(t)+1} - t$ is called the *excess lifetime* at t and $C(t) = t - S_{N(t)}$ is called the *current lifetime* (or *age*) at t (Grimmett and Stirzaker 1992). That is, E(t) is the time which elapses before the next arrival. C(t) is the elapsed time since the last arrival. It is known that $P(C(t) \le t) = 1$ and P(C(t) = t) = 1 - F(t). Since E(t - y) > y if and only if no arrivals occur in (t - y, t], $P(C(t) \le y) = P(E(t - y) \le y)$, y < t.

Renewal Reward Processes

Consider a renewal process $\{N(t), t \ge 0\}$ having interarrival times $X_n, n \ge 1$ with distribution function F, and suppose that each time a renewal occurs we receive a reward. Let R_n denote the reward earned at the time of the *n*th renewal. It is assumed that the $R_n, n \ge 1$, are independent and identically distributed. However, it is allowed for the possibility that R_n may depend on X_n , the length of the *n*th renewal interval, and so it is assumed that the pairs $(X_n, R_n), n \ge 1$, are independent and identically distributed. We consider $\{R(t), t \ge 0\}$ where $R(t) = \sum_{n=1}^{N(t)} R_n$. Then, R(t) represents the total reward earned by time t. If $E(|R|) < \infty$ and $E(X) < \infty$, then as $t \to \infty$,

- \mathbf{i} $R(t)/t \to E(R)/E(X)$ with probability 1,
- ii) $E(R(t))/t \to E(R)/E(X)$

where $E(R) = E(R_n)$ and $E(X) = E(X_n)$ (Ross 1983). This theorem is known in the literature as *renewal-reward theorem*.

4. Age and Block Replacement Policies

We will assume for both policies that units fail permanently, independently and that the time required to perform replacement is negligibly small.

Age Replacement Policy

Under an age replacement policy, a unit is replaced upon failure or at age T, whichever comes first. Denote the number of failures in [0,t] by N(t) and let $\{X_i\}_{i=1}^{\infty}$ represent the durations between successive failures with distribution function F. Denote the total number of removals in [0,t] by $N_A(t,T)$ under age replacement policy with replacement interval T, and denote the number of failures in [0,t] by $N'_A(t,T)$ under age replacement policy with replacement policy.

Theorem 4.1. (Karlin and Taylor 1975) $\{N_A(t,T), t \ge 0\}$ is a renewal process with interrenewal time distribution function

$$F_A(x) = \begin{cases} F(x), & x < T \\ 1, & x \ge T \end{cases}$$

and the mean renewal duration is $\int_0^T (1 - F(x)) dx$.

Proof: Let $\{Y_i\}_{i=1}^{\infty}$ denote the durations between successive removals. Then, $Y_i = min\{X_i, T\}$ for i = 1, 2, Since $X_i, i = 1, 2, ...$, are independent identically distributed, $Y_i, i = 1, 2, ...$, are independent identically distributed as well. Thus, $\{N_A(t,T), t \ge 0\}$ is a renewal process. It is clear that $(Y_1 > y) \equiv (X_1 > y, T > y)$. Therefore,

$$P(Y_1 > y) = \begin{cases} \overline{F}(y), & T > y\\ 0, & T \le y \end{cases}$$

is acquired. Hence,

Also

$$F_A(x) = \begin{cases} F(x), & x < T \\ 1, & x \ge T \end{cases}$$

, $E(Y_1) = \int_0^\infty (1 - F_A(y)) \, dy = \int_0^T (1 - F(x)) \, dx.$

Theorem 4.2. (Barlow and Proschan 1975) $\{N'_A(t,T), t \ge 0\}$ is a renewal process with interrenewal time distribution function,

$$G_A(x) = 1 - \overline{F}(T)^n \overline{F}(x - nT), \quad nT \le x \le (n+1)T, \quad n = 0, 1, \dots$$

and the renewal duration has the expectation $\frac{1}{F(T)} \int_0^T (1 - F(x)) dx$.

Proof: Let $\{Y_i\}_{i=1}^{\infty}$ denote the durations between unplanned failures under age replacement policy. According to the process, from the time that an unplanned failure occurs, the process continues probabilistically in the same way. Therefore, $\{Y_i\}_{i=1}^{\infty}$ random variables are independent identically distributed.

$$P(Y_1 > x) = P(X_1 > x) = \overline{F}(x), \text{ for } 0 < x < T,$$

$$P(Y_1 > x) = P(X_1 > T, X_2 > x - T) = \overline{F}(T) \overline{F}(x - T), \text{ for } T < x < 2T,$$

$$P(Y_1 > x) = P(X_1 > T, X_2 > T, X_3 > x - 2T) = \overline{F}(T)\overline{F}(T)\overline{F}(T)\overline{F}(x - 2T), \text{ for } 2T < x < 3T.$$

In general it can be written as

$$P(Y_1 > x) = \overline{F}(T)^n \overline{F}(x - nT), \quad nT \le x \le (n+1)T, \quad n = 0, 1, \dots$$

It follows that

$$G_A(x) = 1 - \overline{F}(T)^n \overline{F}(x - nT), \quad nT \le x \le (n+1)T, \quad n = 0, 1, \dots$$

Also, the expectation of Y_1 is

$$\begin{split} E(Y_1) &= \int_0^T \overline{F}(x) \, dx + \int_T^{2T} \overline{F}(T) \overline{F}(x-T) \, dx + \int_{2T}^{3T} \overline{F}(T)^2 \, \overline{F}(x-2T) \, dx + \dots \\ &= \sum_{n=0}^\infty \int_{nT}^{(n+1)T} \overline{F}(T)^n \, \overline{F}(x-nT) \, dx \\ &= \sum_{n=0}^\infty \overline{F}(T)^n \, \int_0^T \overline{F}(y) \, dy = \frac{1}{F(T)} \, \int_0^T \overline{F}(y) \, dy \, . \end{split}$$

An Optimal Policy for Age Replacement

We assume an age replacement which is basic for controlling items that are subject to stochastic breakdowns. A cost of $c_p > 0$ is incurred for each planned replacement and a cost of c_f for each failure replacement where $c_f > c_p$. Under age replacement policy, it is obvious that the cost incurred during one replacement cycle is the random variable Y which is

$$Y = \begin{cases} c_f, & X_1 < T\\ c_p, & X_1 \ge T. \end{cases}$$
$$E(Y) = c_f F(T) + c_p (1 - F(T)).$$

Hence, by the renewal-reward theorem,

the long - run average cost per unit time =
$$\frac{c_p + (c_f - c_p)F(T)}{\int_0^T (1 - F(x)) \, dx}$$

with probability 1. By putting the derivative of the average cost function equal to zero, it is verified that the minimizing value of T is the unique solution to the equation

$$r(T) \int_0^T (1 - F(x)) \, dx - F(T) = \frac{c_p}{c_f - c_p}$$

where r(T) is the failure rate function defined by $r(T) = f(T)/\overline{F}(T)$, and it is assumed that this function is continuous and strictly increasing to infinity (Tijms 1995).

Denote the long-run average cost per unit time for the age replacement rule with limit T by g(T) and let T^* be the optimal value of T. Then,

$$g(T^*) = \frac{c_p + (c_f - c_p F(T^*))}{\int_0^{T^*} (1 - F(x)) \, dx}$$

and

$$r(T^*) \int_0^{T^*} (1 - F(x)) \, dx - F(T^*) = \frac{c_p}{c_f - c_p}.$$

Thus,

$$g(T^*) = \frac{c_p + (c_f - c_p)F(T^*)}{\frac{c_p}{c_f - c_p} \frac{1}{r(T^*)} + \frac{F(T^*)}{r(T^*)}}$$
$$= (c_f - c_p)r(T^*).$$

Block Replacement Policy

Under a block replacement policy, a unit is replaced by a new one upon failure and upon scheduled times $T, 2T, \ldots$. There is always a replacement at the scheduled times regardless of the age of the item in use. Denote the total number of removals in [0, t] by $N_B(t, T)$ under block replacement policy with replacement interval T, and denote the number of failures in [0, t] by $N'_B(t, T)$ under block replacement policy with replacement interval T.

While $\{N_A(t,T), t \ge 0\}$ and $\{N'_A(t,T), t \ge 0\}$ are renewal processes, $\{N_B(t,T), t \ge 0\}$ is not a renewal process.

Let the times between removals be $\{Y_i\}_{i=1}^{\infty}$ in $\{N_B(t,T), t \ge 0\}$. Then,

$$\begin{split} Y_1 &= \min\{X_1, T\} \\ Y_2 &= \begin{cases} \min\{X_2, T\}, & Y_1 = T \\ \min\{X_2, T - X_1\}, & Y_1 = X_1. \end{cases} \end{split}$$

It is easily obtained that

$$F_{Y_1}(y) = \begin{cases} F(y), & y < T \\ 1, & y \ge T \end{cases}$$

and

$$F_{Y_2}(y) = \begin{cases} F(y) + (1 - F(y))(F(T) - F(T - y)), & y < T \\ 1, & y \ge T. \end{cases}$$

As we see from the distribution functions of Y_1 and Y_2 , they are not identically distributed. Hence, $\{N_B(t,T), t \ge 0\}$ process cannot be a renewal process. Furthermore, the counting process $\{N'_B(t,T), t \ge 0\}$ is also not a renewal process.

An Optimal Policy for Block Replacement

Assume that the cost structure is the same as in the age replacement policy. The stochastic process describing the age of the item in use is regenerative. The length of one cycle is T. Further,

$$E(Y) = c_p + c_f M(T)$$

where the random variable Y is the cost that is incurred during one replacement cycle and M denotes the renewal function associated with the lifetime distribution function F. This follows by noting that the number of renewals up to time T in the renewal process generated by the lifetimes $X_1, X_2, ...$ is nothing else than the number of failure replacements up to time T. Hence, for the block replacement with parameter T,

the long - run average cost per unit time =
$$\frac{1}{T} \{c_p + c_f M(T)\}$$

with probability 1 (Tijms 1995). If the T value which makes this function minimum exists, then the policy that we use with this T value is the optimal policy for the block replacement.

Replacement Comparisons

It is useful to compare block replacement with age replacement, using replacement interval T for both of them. For example, block replacement is more wasteful since more unfailed components are removed than under age replacement. Under the IFR assumption, the expected number of failures will be less under block replacement. The following theorem, true for all distributions, is intuitively obvious.

Theorem 4.3. $P(N(t) \ge n) \le P(N_A(t,T) \ge n) \le P(N_B(t,T) \ge n)$ for all $t \ge 0, n = 0, 1, ...$.

Proof: Let $\{X_i\}_{i=1}^{\infty}$ represent the durations between successive failures. Let $S_n^A(S_n^B)$ denote the time of the *n*th removal under age (block) replacement policy.

 $Y_1^B = \min\{X_1, T\},\$

 $Y_2^B = min\{X_2, \alpha_2\}, \ 0 \le \alpha_2 \le T,$

 $Y_3^B = min\{X_3, \alpha_3\}, \ 0 \le \alpha_3 \le T,$

generally, we have $Y_B^n = \min\{X_n, \alpha_n\}$, for $0 \le \alpha_n \le T$ under block replacement policy, where α_n is the remaining life to a planned renewal after (n-1)th removal. Under age replacement policy; $Y_n^A = \min\{X_n, T\}$ for n = 1, 2, Then,

$$S_n^A = S_{n-1}^A + \min\{X_n, T\}, \qquad S_n^B = S_{n-1}^B + \min\{X_n, \alpha_n\}$$

Since initially $S_1^A = S_1^B$, $S_n \ge S_n^A \ge S_n^B$. Thus, the proof is completed (Barlow and Proschan 1963).

Let X and Y be any two random variables. It is said that the random variable X is stochastically larger than the random variable Y, written $X \ge_{st} Y$, if $P(X > a) \ge P(Y > a)$ for all a. Also from Theorem 4.3 we have $N(t) \le_{st} N_A(t,T) \le_{st} N_B(t,T)$ which means that the number of removals in [0, t] interval under block replacement policy is stochastically bigger than the number of removals under age replacement policy.

Corollary: $M(t) \leq E(N_A(t,T)) \leq E(N_B(t,T)).$

This corollary is an immediate consequence of Theorem 4.3.

It is shown by Barlow and Proschan (1963) that if F is IFR, then

$$P(N(t) \ge n) \ge P(N'_{A}(t,T) \ge n) \ge P(N'_{B}(t,T) \ge n)$$

for $t \ge 0$ and n = 0, 1, ... which means $N(t) \ge_{st} N'_A(t, T) \ge_{st} N'_B(t, T)$. Equality is attained for the exponential distribution $F(x) = 1 - e^{-x/\mu_1}$ where μ_1 denotes the mean of F. As a consequence of this, we have $M(t) \ge E(N'_A(t,T)) \ge E(N'_B(t,T))$.

Theorem 4.4. $N(t) \geq_{st} N'_A(t,T)$ for all $t \geq 0, T \geq 0 \Leftrightarrow F$ is NBU (Barlow and Proschan, 1975).

Theorem 4.4 states that the class of NBU distributions is the largest class for which age replacement diminishes stochastically the number of failures experienced in any particular time interval [0, t]. In this sense, the NBU class of distributions is a natural class to consider in age replacement.

Let F be IFR. For fixed T > 0, we know that $\{N'_A(t,T), t \ge 0\}$ is a renewal process with underlying distribution function $F_A(t;T) = 1 - \overline{F}(T)^n \overline{F}(x - nT)$, where $nT \le x \le (n+1)T$ for n = 0, 1, ... By differentiating $F_A(t;T)$ with respect to T, it is verified that $F_A(t;T)$ is increasing in $T \ge 0$ for fixed $t \ge 0$. Hence $P(N'_A(t,T) \ge n) = F_A^{n*}(t;T)$ is increasing in $T \ge 0$ for fixed $t \ge 0$, where $F_A^{n*}(t;T)$ is the *n*-fold Stieltjes convolution of $F_A(t;T)$ with itself (Barlow and Proschan 1975). Thus, under age replacement policy with an IFR failure distribution, the number of failures observed in any interval [0, t] increases stochastically as the replacement interval T increases. Also, the direct contrary is true (Marshall and Proschan 1972).

Lemma 4.1. Consider two policies such that the planned replacements occur at fixed time points $\{t_1, t_2, ...\}$ under policy 1, and at the time points $\{t_1, t_2, ...\} \cup \{t_0\}$ under policy 2, where $0 < t_1 < t_2, ...$. Let $N_i(t)$ be the number of failures in [0, t] under policy i, i = 1, 2. Then $N_1(t) \geq_{st} N_2(t)$ for each $t \geq 0$ if and only if the underlying life distribution function F is NBU (Barlow and Proschan 1975).

Theorem 4.5. $N(t) \geq_{st} N'_B(t,T)$ for all $t \geq 0, T \geq 0 \Leftrightarrow F$ is NBU (Barlow and Proschan 1975).

Theorem 4.5 states that the class of NBU distributions is the largest class for which block replacement diminishes stochastically the number of failures in any particular time interval [0, t], $0 < t < \infty$.

We know that under age replacement policy, the number of failures observed in any interval [0, t] increases stochastically as the replacement interval T increases if and only if F is IFR. Under block replacement policy, it is shown by Shaked and Zhu (1992) that the stochastic increasingness of $N'_B(t,T)$ in $T \ge 0$, for each fixed $t \ge 0$, is a sufficient condition for F to be IFR, but it is not a necessary condition. Suppose that $\{N'_B(t,T)\}$ increases stochastically as the replacement interval T increases for fixed $t \ge 0$. Hence, $P(N'_B(t,T_2) \ge n) \ge P(N'_B(t,T_1) \ge n)$ for $T_1 \le T_2$. We choose n = 1, then it is clear that

$$P(N'_B(t, T_2) = 0) = 1 - P(N'_B(t, T_2) \ge 1)$$

$$\leq 1 - P(N'_B(t, T_1) \ge 1)$$

$$= P(N'_B(t, T_1) = 0).$$

Let $T_1 \leq T_2 \leq 2T_1$ and we choose t such that $T_1 \leq T_2 \leq t \leq 2T_1$. Since $P(N'_B(t,T_1)=0) = \overline{F}(T_1)\overline{F}(t-T_1)$ and $P(N'_B(t,T_2)=0) = \overline{F}(T_2)\overline{F}(t-T_2)$, we have

(3)
$$\overline{F}(T_1)\overline{F}(t-T_1) \ge \overline{F}(T_2)\overline{F}(t-T_2).$$

We need to show $\overline{F}(x+\Delta)/\overline{F}(x) \ge \overline{F}(y+\Delta)/\overline{F}(y)$ for any given $y \ge x \ge 0$ and $\Delta \ge 0$.

i) Let $\Delta > y - x$. Then, we take $T_1 = x + \Delta$, $T_2 = y + \Delta$ and $t = x + y + \Delta$. Then, T_1 , T_2 and t hold $0 \le T_1 \le T_2 \le t \le 2T_1$. Since $t - T_1 = y$, $t - T_2 = x$, $T_1 = x + \Delta$ and $T_2 = y + \Delta$, by (3) we have $\overline{F}(x + \Delta)/\overline{F}(x) \ge \overline{F}(y + \Delta)/\overline{F}(y)$.

ii) Let $\Delta \leq y - x$. Then, we take $T_1 = y$, $T_2 = y + \Delta$ and $t = x + y + \Delta$. Then, T_1 , T_2 and t hold $0 \leq T_1 \leq T_2 \leq t \leq 2T_1$. Since $t - T_1 = x + \Delta$, $t - T_2 = x$, $T_1 = y$ and $T_2 = y + \Delta$, by (3) we have $\overline{F}(x + \Delta)/\overline{F}(x) \geq \overline{F}(y + \Delta)/\overline{F}(y)$. Hence, if $N'_B(t,T) \uparrow_{st}$ in $T \ge 0$ for each fixed $t \ge 0$, then F is IFR.

Now, some comparisons are given for age and block replacement policies under the assumption that the underlying distribution function F is NBUL or NWUL due to Yue and Cao (2001).

Let $\{X_i\}_{i=1}^{\infty}$ and $\{Y_i\}_{i=1}^{\infty}$ denote the interrenewal times for the renewal processes $\{N(t), t \ge 0\}$ and $\{N'_A(t), t \ge 0\}$. It is obvious that

$$X_1 \leq_L (\geq_L) Y_1 \Leftrightarrow X_i \leq_L (\geq_L) Y_i \qquad i = 1, 2, \dots$$

Theorem 4.6. F is NBUL(NWUL) $\Leftrightarrow X_1 \leq_L (\geq_L) Y_1$.

Proof: We know from Theorem 4.2 that $P(Y_1 > x) = \overline{F}(T)^n \overline{F}(x-nT), \quad nT \le x \le (n+1)T, \quad n = 0, 1, \dots$. Thus,

$$\int_0^\infty e^{-sx} P(Y_1 > x) \, dx = \sum_{n=0}^\infty \int_{nT}^{(n+1)T} e^{-sx} \overline{F}(T)^n \, \overline{F}(x - nT) \, dx$$
$$= \sum_{n=0}^\infty [\overline{F}(T) \, e^{-sT}]^n \, \int_0^T e^{-sx} \, \overline{F}(x) \, dx$$
$$= \int_0^T e^{-sx} \, \overline{F}(x) \, dx \, / [1 - e^{-sT} \, \overline{F}(T)].$$

From the definition of Laplace ordering we have

$$X_1 \leq_L (\geq_L) Y_1 \Leftrightarrow \int_0^\infty e^{-sx} \overline{F}(x) \, dx \leq (\geq) \int_0^\infty e^{-st} P(Y_1 > x) \, dx.$$

Hence,

$$X_1 \leq_L (\geq_L) Y_1 \Leftrightarrow \int_0^\infty e^{-sx} \overline{F}(x) \, dx \leq (\geq) \int_0^T e^{-sx} \overline{F}(x) \, dx \, / [1 - e^{-sT} \overline{F}(T)].$$

This is equivalent to F is NBUL(NWUL) (Yue and Cao 2001).

Let $\{N_1(t), t \ge 0\}$ and $\{N_2(t), t \ge 0\}$ be two counting processes such that

$$\int_0^\infty e^{-st} P(N_1(t) \ge n) \, dt \le \int_0^\infty e^{-st} P(N_2(t) \ge n) \, dt$$

for all s > 0 and n = 0, 1, ... Then, $\{N_1(t), t \ge 0\}$ is said to be smaller than $\{N_2(t), t \ge 0\}$ in Laplace order and denoted as $N_1(t) \le_L N_2(t)$.

Theorem 4.7. Let $\{N_1(t), t \ge 0\}$ and $\{N_2(t), t \ge 0\}$ be two renewal processes. X_i^1 and X_i^2 denote the duration between (i-1)th and *i*th renewal for $\{N_1(t), t \ge 0\}$ and $\{N_2(t), t \ge 0\}$, respectively. Then, $N_2(t) \le_L N_1(t)$ if and only if $X_i^1 \leq_L X_i^2, \, i=1,2,\ldots$.

Proof: " \Rightarrow ":

$$\int_0^\infty e^{-st} P(N_1(t) \ge 1) \, dt \ge \int_0^\infty e^{-st} P(N_2(t) \ge 1) \, dt$$

or equivalently

$$\int_0^\infty e^{-st} P(N_1(t) = 0) \, dt \le \int_0^\infty e^{-st} P(N_2(t) = 0) \, dt \, .$$

Observing that $P(N_k(t) = 0) = P(X_i^k > t)$ for k = 1, 2 and i = 1, 2, ... we have

$$\int_0^\infty e^{-st} P(X_i^1 > t) \, dt \le \int_0^\infty e^{-st} P(X_i^2 > t) \, dt \, .$$

Thus, $X_i^1 \leq_L X_i^2$, i = 1, 2,

" \Leftarrow " : We have $\sum\limits_{i=1}^n X_i^1 \leq_L \sum\limits_{i=1}^n X_i^2$ (Alzaid, Kim and Proschan 1991), or equivalently

$$\int_0^\infty e^{-st} P(\sum_{i=1}^n X_i^1 > t) \, dt \le \int_0^\infty e^{-st} P(\sum_{i=1}^n X_i^2 > t) \, dt.$$

Hence,

$$\int_0^\infty e^{-st} P(\sum_{i=1}^n X_i^1 \le t) \, dt \ge \int_0^\infty e^{-st} P(\sum_{i=1}^n X_i^2 \le t) \, dt.$$

It follows that

$$\int_0^\infty e^{-st} P(N_1(t) \ge n) \, dt \ge \int_0^\infty e^{-st} P(N_2(t) \ge n) \, dt$$

for all s > 0 and $n = 1, 2, \dots$. Thus, $N_2(t) \leq_L N_1(t)$ (Yue and Cao 2001).

As a consequence of Theorem 4.6 and Theorem 4.7 we have;

$$N(t) \ge_L (\le_L) N'_A(t,T) \Leftrightarrow F \text{ is NBUL(NWUL)}$$

which states that the age replacement diminishes (increases) , in the sense of Laplace order, the number of failures in any particular time interval [0, t], $0 < t < \infty$, if and only if F is NBUL(NWUL).

Lemma 4.2. Let planned replacements occur at fixed time points $\{0 < t_1 < t_2 < ...\}$ under policy 1, and at time points $\{0 < t_1 < t_2 < ...\} \cup \{t_0\}$ under policy 2. Let $N_i(t)$ be the number of failures in [0, t] under policy i, i = 1, 2.

Then, $N_1(t) \ge_L (\le_L)N_2(t)$ for each $t \ge 0$ if and only if the underlying life distribution F is NBUL(NWUL).

The proof of Lemma 4.2 is similar to that of Lemma 4.1. The next theorem given by Yue and Cao (2001) states that the block replacement diminishes (increases), in the sense of Laplace order, the number of failures experienced in any particular time interval [0, t], $0 < t < \infty$, if and only if F is NBUL(NWUL).

Theorem 4.8. $N(t) \ge_L (\le_L) N'_B(t,T), t \ge 0, T \ge 0 \Leftrightarrow F$ is NBUL(NWUL).

Proof: " \Leftarrow ": Let planned replacements occur at fixed time points $\{0, T, ..., (i-1)T\}$ under policy i, i = 1, 2, Let $N_i(t)$ be the number of failures in [0, t] under policy i, i = 1, 2, It follows from Lemma 4.2 that $N(t) = N_1(t) \ge_L N_2(t) \ge_L ... \ge_L N_k(t) \ge_L ...$. As $k \to \infty$, we have $N(t) \ge_L N'_B(t, T)$.

" \Rightarrow " : It is clear that

$$\int_{0}^{\infty} e^{-st} P(N(t) \ge 1) \, dt \ge \int_{0}^{\infty} e^{-st} P(N'_{B}(t, T) \ge 1) \, dt.$$

Hence,

(4)
$$\int_{0}^{\infty} e^{-st} P(N(t) = 0) dt \le \int_{0}^{\infty} e^{-st} P(N'_{B}(t, T) = 0) dt.$$

Observing that

$$P(N'_{B}(t,T) = 0) = P(X_{1} > T, ..., X_{k} > T, X_{k+1} > t - kT)$$

= $\overline{F}(T)^{k} \overline{F}(t - kT)$

for $kT \leq t < (k+1)T, \, k=0,1,\ldots$. Then,

$$\int_0^\infty e^{-st} P(N'_B(t,T)=0) dt = \sum_{k=0}^\infty \int_{kT}^{(k+1)T} e^{-st} \overline{F}(T)^k \overline{F}(t-kT) dt$$
$$= \sum_{k=0}^\infty [\overline{F}(T) e^{-sT}]^k \int_0^T e^{-st} \overline{F}(t) dt$$
$$= \int_0^T e^{-st} \overline{F}(t) dt/(1-e^{-sT} \overline{F}(T)).$$

Noting that $\int_0^\infty e^{-st} P(N(t) = 0) dt = \int_0^\infty e^{-st} \overline{F}(t) dt$ and from (4) we obtain

$$\int_0^\infty e^{-st} \,\overline{F}(t) \, dt \le \frac{\int_0^T e^{-st} \,\overline{F}(t) \, dt}{1 - e^{-sT} \,\overline{F}(T)}$$

Thus, F is NBUL. The proof in NWUL case is similar.

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