Selçuk J. Appl. Math.
Vol. 12. No. 2. pp. 21-33, 2011

Selçuk Journal of Applied Mathematics

A Subset of the Space of the χ^2 Sequences[1]

N. Subramanian

Department of Mathematics,SASTRA University, Thanjavur-613 401, India
e-mail: nsmaths@yahoo.com

Received Date: July 23, 2010
Accepted Date: January 5, 2011

Abstract. Let χ^2 denote the space of all Pringsheim sense double gai sequences. Let Λ^2 denote the space of all Pringsheim sense double analytic sequences. This paper is devoted to a study of the general properties of Sectional space χ_s^2 of χ^2.

Key words: Double gai sequence, double analytic sequence, Sectional sequence spaces.
2000 Mathematics Subject Classification: 40A05, 40C05, 40D05.

1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.
We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [3]. Later on, they were investigated by Hardy [5], Moricz [6], Moricz and Rhoades [7], Basarir and Solankan [2], Tripathy [8], Colak and Turkmenoglu [4], Turkmenoglu [9], and many others.

Let us define the following sets of double sequences:

[1]Dedicated to my beloved Professor D. Jeyamani, Department of Mathematics, SBK College, Aruppukottai-626 101, India, a committed teacher, on his retirement from service but not from teaching

$$\mathcal{M}_u(t) := \left\{ (x_{mn}) \in w^2 : sup_{m,n \in N} |x_{mn}|^{t_{mn}} < \infty \right\},$$
$$\mathcal{C}_p(t) := \left\{ (x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - |^{t_{mn}} = 1 \text{ for some } \in \mathbb{C} \right\},$$
$$\mathcal{C}_{0p}(t) := \left\{ (x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1 \right\},$$
$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},$$
$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [11,12] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [13] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [14] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [15] and Mursaleen and Edely [16] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x. More recently, Altay and Basar [17] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Basar and Sever [18] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [19] have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations.
We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p \tag{1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see[1]).

A sequence $x = (x_{mn})$is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2.A sequence $x = (x_{mn})$ is called double entire sequence if $|x_{mn}|^{1/m+n} \to 0$ as $m, n \to$

∞. The double entire sequences will be denoted by Γ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)!\,|x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{all\,finite\,sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a 1 in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous.

If X is a sequence space, we give the following definitions:

(i) X'= the continuous dual of X;

(ii) $X^\alpha = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \text{for each}\, x \in X\right\}$;

(iii) $X^\beta = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn}\, \text{is convegent, for each}\, x \in X\right\}$;

(iv) $X^\gamma = \left\{a = (a_{mn}) : sup_{mn} \geq 1 \left|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, \text{ for each } x \in X\right\}$;

(v) let X be an $FK-$space $\supset \phi$; then $X^f = \left\{f(\Im_{mn}) : f \in X'\right\}$;

(vi) $X^\delta = \left\{a = (a_{mn}) : sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty, \text{for each}\, x \in X\right\}$;

$X^\alpha.X^\beta, X^\gamma$ are called $\alpha-$ (or Köthe-Toeplitz) dual of X, $\beta-$(or generalized-Köthe-Toeplitz) dual of X, $\gamma-$ dual of X, $\delta-$ dual of X respectively.X^α is defined by Gupta and Kamptan [10]. It is clear that $x^\alpha \subset X^\beta$ and $X^\alpha \subset X^\gamma$, but $X^\alpha \subset X^\gamma$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.
The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [20] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Here w, c, c_0 and ℓ_∞ denote the classes of all, convergent,null and bounded sclar valued single sequences respectively. The above spaces are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k\geq 1} |\Delta x_k|$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$
We recall that cs_0^2 denotes the vector space of all sequences $x = (x_{mn})$ such that $\{\xi_{mn}\}$ is a double null sequence.

2. Definitions and Preliminaries

A double sequence $x = (x_{mn})$ is called convergent (with limit L) if and only if for every $\epsilon > 0$ there exists a positive integer $n_0 = n_0(\epsilon)$ such that $|x_{mn} - L| < \epsilon$, for all $m, n \geq n_0$. We write $x_{mn} \to L$ or $lim_{m,n\to\infty} x_{mn} = L$ if (x_{mn}) is convergent to L. The limit L is called double limit or Pringsheim sense limit.
A sequence $x = (x_{mn})$ is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all Pringsheim sense double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called Pringsheim sense double entire sequence if $|x_{mn}|^{1/m+n} \to 0 \,\text{as}\, m, n \to \infty$. The double entire sequences will be denoted by Γ^2. The space Λ^2 and Γ^2 is a metric space with the metric

$$d(x, y) = sup_{mn} \left\{|x_{mn} - y_{mn}|^{1/m+n} : m, n : 1, 2, 3, \cdots\right\} \tag{2}$$

for all $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in Γ^2.
A sequence $x = (x_{mn})$ is called Pringsheim sense double gai sequence if $((m+n)! \, |x_{mn}|)^{1/m+n} \to 0 \,\text{as}\, m, n \to \infty$. The double gai sequences will be denoted by χ^2. The space χ^2 is a metric space with the metric

$$\widetilde{d}(x, y) = sup_{mn} \left\{((m+n)! \, |x_{mn} - y_{mn}|)^{1/m+n} : m, n : 1, 2, 3, \cdots\right\} \tag{3}$$

forall $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in χ^2.

Let $\chi_s^2 = \left\{x = (x_{mn}) : \xi : (\xi_{mn}) \in \chi^2\right\}$
where $\xi_{mn} = \alpha_{11} + \alpha_{22} + \cdots + \alpha_{mn}$ for $m, n = 1, 2, 3, \cdots .Here$
$\alpha_{11} = x_{11} + x_{12} + \cdots + x_{1n};$
$\alpha_{22} = x_{21} + x_{22} + \cdots + x_{2n};$
$\vdots$
$\alpha_{mn} = x_{m1} + x_{m2} + \cdots + x_{mn}.$ and
$\Lambda^2 = \left\{y = (y_{mn}) : \eta : (\eta_{mn}) \in \Lambda^2\right\}$
where $\eta_{mn} = \beta_{11} + \beta_{22} + \cdots + \beta_{mn}$ for $m, n = 1, 2, 3, \cdots$. Here
$\beta_{11} = y_{11} + y_{12} + \cdots + y_{1n};$
$\beta_{22} = y_{21} + y_{22} + \cdots + y_{2n};$
$\vdots$

$\beta_{mn} = y_{m1} + y_{m2} + \cdots + y_{mn}.$

The space Λ_s^2 is a metric space with the metric

(4) $$d(x,y) = sup_{mn} \left\{ |\xi_{mn} - \eta_{mn}|^{1/m+n} : m,n : 1,2,3,\cdots \right\}$$

for all $\xi = \{\xi_{mn}\} \, \text{and} \, \eta = \{\eta_{mn}\} \text{in} \, \Lambda^2.$

The space χ_s^2 is a metric space with the metric

(5) $$\widetilde{d}(x,y) = sup_{mn} \left\{ ((m+n)! \, |\xi_{mn} - \eta_{mn}|)^{1/m+n} : m,n : 1,2,3,\cdots \right\}$$

forall $\xi = \{\xi_{mn}\} \, \text{and} \, \eta = \{\eta_{mn}\} \text{in} \, \chi^2.$

Let $\sigma\left(\chi^2\right)$ denote the vector space of all sequences $x = \{x_{mn}\}$ such that $\left\{ \frac{\xi_{mn}}{(m+n)} \right\}$ is an double gai sequence.

A sequence E is said to be solid (or normal) if $(\lambda_{mn} x_{mn}) \in E$, whenever $(x_{mn}) \in E$ for all sequences of scalars $(\lambda_{mn} = k)$ with $|\lambda_{mn}| \leq 1$.

Remark.
$x = (x_{mn}) \in \sigma\left(\chi^2\right) \Leftrightarrow \left\{ \frac{\alpha_{11} + \alpha_{22} + \cdots + \alpha_{mn}}{m+n} \right\} \in \chi^2.$
$\Leftrightarrow \left| \frac{(m+n)! |\alpha_{11} + \alpha_{22} + \cdots + \alpha_{mn}|}{(m+n)} \right|^{1/m+n} \to 0 \, as \, m,n \to \infty$
$\Leftrightarrow ((m+n)! \, |\alpha_{11} + \alpha_{22} + \cdots + \alpha_{mn}|)^{1/m+n} \to 0 \, as \, m,n \to \infty, \text{because}$
$$(m+n)^{1/m+n} \to 1 \, \text{as} \, m,n \to \infty.$$
$\Leftrightarrow (x_{mn}) \in \chi_s^2$
$\Leftrightarrow \sigma\left(\chi^2\right) \in \chi_s^2.$
In this paper we investigate:
(i) set-inclusion between χ_s^2 and χ^2,
(ii) AK-property possessed by χ_s^2,
(iii)Solidity of χ_s^2 as a linear space,
(iv) $\beta-$ dual of χ_s^2.

3. Main Results

3.1. Proposition. $\chi_s^2 \subset \chi^2$.

Proof. Let $x \in \chi_s^2$
$\Rightarrow \xi \in \chi^2$

(6) $$((m+n)! \, |\xi_{mn}|)^{1/m+n} \to 0 \, as \, m,n \to \infty$$

But $x_{mn} = \xi_{mn} - \xi_{mn+1} - \xi_{m+1n} + \xi_{m+1n+1}$
Hence $((m+n)! \, |x_{mn}|)^{1/m+n} \leq ((m+n)! \, |\xi_{mn}|)^{1/m+n} + \left((m+n)! \, |\xi_{mn+1}|\right)^{1/m+n} +$

$\left((m+n)!\left|\xi_{m+1n}\right|\right)^{1/m+n} + \left((m+n)!\left|\xi_{m+1n+1}\right|\right)^{1/m+n} \rightarrow 0\, as\, m,n \rightarrow \infty$ by using (6)
$\Rightarrow x \in \chi^2$.
$\Rightarrow \chi_s^2 \subset \chi^2$.
Note The above inclusion is strict.

$$\text{Take the sequence } \Im_{mn} = \begin{pmatrix} \frac{1}{(m+n)!}, & 0, & ...0 \\ 0, & 0, & ...0 \\ . & & \\ . & & \\ . & & \\ 0, & 0, & ...0 \end{pmatrix} \in \chi^2. \text{We have}$$

$\alpha_{11} = \frac{1}{(m+n)!} + 0 + 0 + \cdots + 0 = \frac{1}{(m+n)!}$
$\alpha_{22} = 0 + 0 + \cdots + 0 = 0$
$\alpha_{33} = 0 + 0 + \cdots + 0 = 0$
$\vdots$
$\alpha_{mn} = 0 + 0 + 0 + \cdots + 0 = 0$
$\rightarrow mn^{th} - row \leftarrow$
and so on.
Now $\left((m+n)!\left|\xi_{mn}\right|\right)^{1/m+n} = 1$ for all m,n. Hence $\left\{\left((m+n)!\left|\xi_{mn}\right|\right)^{1/m+n}\right\}$ does not tend to zero as $m,n \rightarrow \infty$. So $\Im_{mn} \notin \chi_s^2$. Thus the inclusion $\chi_s^2 \subset \chi^2$ is strict. This completes the proof.

3.2. Proposition. χ_s^2 has AK property.

Proof. Let $x = (x_{mn}) \in \chi_s^2$ and take the $[mn]^{th}$ sectional sequence we have

$$x^{[rs]} = \begin{pmatrix} x_{11}, & x_{12}, & ...x_{1n}, & 0 \\ . & & & \\ . & & & \\ . & & & \\ x_{m1}, & x_{m2}, & ...x_{mn}, & 0 \\ 0, & 0, & ...0, & 0 \end{pmatrix}, \text{for } m \geq r, n \geq s. \text{Hence}$$

$d\left(x, x^{[r,s]}\right) = \sup_{mn}\left\{\left((m+n)!\left|\xi_{mn} - \xi_{mn}^{[rs]}\right|\right)^{1/m+n} : m \geq r, n \geq s\right\} \rightarrow 0$ as $[r,s] \rightarrow \infty$. Therefore $x^{[rs]} \rightarrow x \in \chi_s^2$ as $r,s \rightarrow \infty$. Thus χ_s^2 has AK. This completes the proof.

3.3. Proposition. χ_s^2 is a linear space over field $\mathbb{C}$ of compleex numbers.

Proof. Let $x = (x_{mn})$ and $y = (y_{mn})$ belong to χ_s^2. Let $\alpha, \beta \in \mathbb{C}$. Then $\xi = (\xi_{mn}) \in \chi^2$ and $\eta = (\eta_{mn}) \in \chi^2$. But χ^2 is a linear space. Hence $\alpha\xi + \beta\eta \in \chi^2$. Consequently $\alpha x + \beta y \in \chi_s^2$. Therefore χ_s^2 is linear. This completes the proof.

3.4. Proposition. χ_s^2 is solid.

Proof. Let $|x_{mn}| \leq |y_{mn}|$ with $y = (y_{mn}) \in \chi_s^2$. So $|\xi_{mn}| \leq |\eta_{mn}|$ with $\eta = (\eta_{mn}) \in \chi^2$. But χ^2 is solid. Hence $\xi = (\xi_{mn}) \in \chi^2$. Therefore $x = (x_{mn}) \in \chi_s^2$. Hence χ_s^2 is solid. This completes the proof.

3.5. Proposition. The $\beta-$ dual space of χ_s^2 is Λ^2.

Proof. *Step 1.* By Proposition 3.1, we have $\chi_s^2 \subset \chi^2$. Hence $\left(\chi^2\right)^\beta \subset \left(\chi_s^2\right)^\beta$. But $\left(\chi^2\right)^\beta = \Lambda^2$. Therefore

$$\Lambda^2 \subset \left(\chi_s^2\right)^\beta. \tag{7}$$

Step 2. Next we show that $\left(\chi_s^2\right)^\beta \subset \Lambda^2$. Let $y = (y_{mn}) \in \left(\chi_s^2\right)^\beta$. Consider $f(x) = \sum_{m=1}^\infty \sum_{n=1}^\infty x_{mn} y_{mn}$ with $x = (x_{mn}) \in \chi_s^2$
$x = [(\Im_{mn} - \Im_{mn+1}) - (\Im_{m+1n} - \Im_{m+1n+1})]$

$$= \begin{pmatrix} 0, & 0, & ...0, & 0, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \\ . \\ . \\ . \\ 0, & 0, & ...\frac{1}{(m+n)!}, & \frac{-1}{(m+n)!}, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \end{pmatrix} - \begin{pmatrix} 0, & 0, & ...0, & 0, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \\ . \\ . \\ . \\ 0, & 0, & ...0, & 0, & ... & 0 \\ 0, & 0, & ...\frac{1}{(m+n)!}, & \frac{-1}{(m+n)!}, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \end{pmatrix}$$

$$\left\{((m+n)!\,|x_{mn}|)^{1/m+n}\right\} = \begin{pmatrix} 0, & 0, & ...0, & 0, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \\ . \\ . \\ . \\ 0, & 0, & ...\frac{1}{(m+n)!}, & \frac{-1}{(m+n)!}, & ... & 0 \\ \\ 0, & 0, & ...\frac{-1}{(m+n)!}, & \frac{1}{(m+n)!}, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \end{pmatrix}$$. Hence converges to zero.

Therefore $[(\Im_{mn} - \Im_{mn+1}) - (\Im_{m+1n} - \Im_{m+1n+1})] \in \chi_s^2$.

Hence $d\left((\Im_{mn} - \Im_{mn+1}) - (\Im_{m+1n} - \Im_{m+1n+1}), 0\right) = 1$. But

$|y_{mn}| \leq \|f\| \, d\left(\left(\Im_{mn} - \Im_{mn+1}\right) - \left(\Im_{m+1n} - \Im_{m+1n+1}\right), 0\right) \leq \|f\| \cdot 1 < \infty$ for each m, n. Thus (y_{mn}) is a double bounded sequence and hence an double analytic sequence. In other words $y \in \Lambda^2$. But $y = (y_{mn})$ is arbitrary in $\left(\chi^2_s\right)^{\beta}$. Therefore

$$\left(\chi^2_s\right)^{\beta} \subset \Lambda^2. \tag{8}$$

From (7) and (8) we get $\left(\chi^2_s\right)^{\beta} = \Lambda^2$. This completes the proof.

3.6. Proposition. $\Lambda^2 \subset \left(\chi^2_s\right)^{\beta} \subset \Lambda^2(\Delta)$.

Proof. *Step 1.* By Proposition 3.1, we have $\chi^2_s \subset \chi^2$. Hence $\left(\chi^2\right)^{\beta} \subset \left(\chi^2_s\right)^{\beta}$. But $\left(\chi^2\right)^{\beta} = \Lambda^2$. Therefore

$$\Lambda^2 \subset \left(\chi^2_s\right)^{\beta}. \tag{9}$$

Step 2. Next we show that $\left(\chi^2_s\right)^{\beta} \subset \Lambda^2$. Let $y = (y_{mn}) \in \left(\chi^2_s\right)^{\beta}$. Consider $f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{mn} y_{mn}$ with $x = (x_{mn}) \in \chi^2_s$
$x = \left[\left(\Im_{mn} - \Im_{mn+1}\right) - \left(\Im_{m+1n} - \Im_{m+1n+1}\right)\right]$

$$= \begin{pmatrix} 0, & 0, & \ldots 0, & 0, & \ldots & 0 \\ 0, & 0, & \ldots 0, & 0, & \ldots & 0 \\ \cdot \\ \cdot \\ \cdot \\ 0, & 0, & \ldots \frac{1}{(m+n)!}, & \frac{-1}{(m+n)!}, & \ldots & 0 \\ 0, & 0, & \ldots \frac{-1}{(m+n)!}, & \frac{1}{(m+n)!}, & \ldots & 0 \\ 0, & 0, & \ldots 0, & 0, & \ldots & 0 \end{pmatrix} \text{where, for each fixed } m, n = 1, 2, 3, \cdots$$

$$\Im_{mn} = \begin{pmatrix} 0, & 0, & \ldots 0, & 0, & \ldots & 0 \\ 0, & 0, & \ldots 0, & 0, & \ldots & 0 \\ \cdot \\ \cdot \\ \cdot \\ 0, & 0, & \ldots \frac{1}{(m+n)!}, & 0, & \ldots & 0 \\ 0, & 0, & \ldots 0, & 0, & \ldots & 0 \end{pmatrix}, \frac{1}{(m+n)!} \text{ in the } (mn)^{th} \text{ place and zero's}$$

elsewhere.
Then
$f\left[\left(\Im_{mn} - \Im_{mn+1}\right) - \left(\Im_{m+1n} - \Im_{m+1n+1}\right)\right] = \left[\left(y_{mn} - y_{mn+1}\right) - \left(y_{m+1n} - y_{m+1n+1}\right)\right].$

Hence

$$|(y_{mn}-y_{mn+1})-(y_{m+1n}-y_{m+1n+1})| = \left| \begin{array}{l} f(\Im_{mn}-\Im_{mn+1}) \\ -(\Im_{m+1n}-\Im_{m+1n+1}) \end{array} \right|$$

$$\begin{aligned} |(y_{mn}-y_{mn+1})-(y_{m+1n}-y_{m+1n+1})| &\leq \|f\| \, d\left(\begin{array}{l} (\Im_{mn}-\Im_{mn+1}) \\ -(\Im_{m+1n}-\Im_{m+1n+1}), 0 \end{array} \right) \\ &\leq \|f\| \cdot 1. \end{aligned}$$

So, $\{(y_{mn}-y_{mn+1})-(y_{m+1n}-y_{m+1n+1})\}$ is double bounded sequence. Consequently $\{(y_{mn}-y_{mn+1})-(y_{m+1n}-y_{m+1n+1})\} \in \Lambda^2$. That is $\{y_{mn}\} \in \Lambda^2(\Delta)$. But $y=(y_{mn})$ is Originally in $\left(\chi_s^2\right)^\beta$. Therefore

$$\left(\chi_s^2\right)^\beta \subset \Lambda^2(\Delta). \tag{10}$$

From (9) and (10) we conclude that $\Lambda^2 \subset \left(\chi_s^2\right)^\beta \subset \Lambda^2(\Delta)$. This completes the proof.

3.7. Proposition. $\left(\Lambda^2\right)^\beta = \Lambda^2$.

Proof. *Step 1.* Let $(x_{mn}) \in \Lambda^2$ and let $(y_{mn}) \in \Lambda^2$. Then we get $|y_{mn}|^{1/m+n} \leq M$ for some constant $M>0$.
Also $(x_{mn}) \in \chi^2 \Rightarrow ((m+n)!\,|x_{mn}|)^{1/m+n} \leq \epsilon = \frac{1}{2M}$
$\Rightarrow |x_{mn}| \leq \frac{1}{2^{m+n}M^{m+n}(m+n)!}$.
Hence $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|x_{mn}y_{mn}| \leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|x_{mn}|\,|y_{mn}|$

$< \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{2^{m+n}}\frac{1}{M^{m+n}}M^{m+n}\frac{1}{(m+n)!}$

$< \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{2^{m+n}}\frac{1}{(m+n)!} < \infty$.
Therefore, we get that $(x_{mn}) \in \left(\Lambda^2\right)^\beta$ and so we have

$$\chi^2 \subset \left(\Lambda^2\right)^\beta. \tag{11}$$

Step 2. Let $(x_{mn}) \in \left(\Lambda^2\right)^\beta$. This says that

$$\Rightarrow \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|x_{mn}y_{mn}| < \infty \,\text{for each}\, (y_{mn}) \in \Lambda^2. \tag{12}$$

Assume that $(x_{mn}) \notin \chi^2$, then there exists a sequence of positive integers (m_p+n_p) strictly increasing such that

$$\left|x_{m_p+n_p}\right| > \frac{1}{2^{m_p+n_p}}\frac{1}{(m+n)!}, (p=1,2,3,\cdots)$$

Take

$$y_{m_p,n_p} = 2^{m_p+n_p}\,(m+n)!\,(p=1,2,3,\cdots)$$

and

$$y_{mn} = 0 \text{ otherwise}$$

Then $(y_{mn}) \in \Lambda^2$. But

$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|x_{mn}y_{mn}| = \sum\sum_{p=1}^{\infty}\left|x_{m_pn_p}y_{m_pn_p}\right| > 1+1+1+\cdots.$

We know that the infinite series $1+1+1+\cdots$ diverges. Hence $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|x_{mn}y_{mn}|$ diverges. This contradicts (12). Hence $(x_{mn}) \in \chi^2$. Therefore

$$\left(\Lambda^2\right)^{\beta} \subset \chi^2. \tag{13}$$

From (11) and (13) we get $\left(\Lambda^2\right)^{\beta} = \chi^2$. This completes the proof.

3.9. Proposition. In χ_s^2 weak convergence does not imply strong convergence.

Proof. Assume that weak convergence implies strong convergence in χ_s^2. Then we would have $\left(\chi_s^2\right)^{\beta\beta} = \chi_s^2$. (See [Wilansky [21]]) But $\left(\chi_s^2\right)^{\beta\beta} = \left(\Lambda^2\right)^{\beta} = \Lambda^2$. Thus $\left(\chi_s^2\right)^{\beta\beta} \neq \left(\chi_s^2\right)$. Hence weak convergence does not imply strong convergence in $\left(\chi_s^2\right)$. This completes the proof.

3.1. Definition. Let $\alpha > 0$ be not an integer. Write $s_{\mu\gamma}^{\alpha\beta} = \sum_{m=1}^{\mu}\sum_{n=1}^{\gamma} A_{\mu-m\gamma-n}^{(\alpha-1)(\beta-1)} x_{mn}$, where $A_{pq}^{(\alpha\beta)}$ denotes the binomial coefficient $\frac{(p+\alpha,q+\beta)(p+\alpha-1,q+\beta-1)\cdots(\alpha+1,\beta+1)}{(pq)!}$ Then $(x_{mn}) \in \sigma^{\alpha\beta}\left(\chi^2\right)$ mean that $\left\{\frac{S_{\mu\gamma}^{(\alpha\beta)}}{A_{\mu\gamma}^{(\alpha-1)(\beta-1)}}\right\} \in \chi^2$.

3.10. Proposition. Let $\alpha, \beta > 0$ be a number which is not an integer. Then $\chi^2 \bigcap \sigma^{\alpha\beta}\left(\chi^2\right) = \theta$, where θ denotes the sequence $\begin{pmatrix} 0, & 0, & ...0 \\ 0, & 0, & ...0 \\ . & & \\ . & & \\ . & & \\ 0, & 0, & ...0 \end{pmatrix}$.

Proof. Since $(x_{mn}) \in \sigma^{\alpha\beta}\left(\chi^2\right)$ we have $\left\{\frac{S_{\mu\gamma}^{(\alpha\beta)}}{A_{\mu\gamma}^{(\alpha-1)(\beta-1)}}\right\} \in \chi^2$. This is equivalent to $\left(S_{\mu\gamma}^{(\alpha\beta)}\right) \in \chi^2$. This, in turn, is equivalent to the assertion that $f_{\alpha\beta}(z) =$

$\sum_{\mu=1}^{\infty}\sum_{\gamma=1}^{\infty} S_{\mu\gamma}^{(\alpha\beta)} z^{(\mu-1,\gamma-1)}$ is an integral function. Now $f_{\alpha\beta}(z) = \frac{f(z)}{(1-z)^{\alpha\beta}}$. Since $\alpha\beta$ is not an integer, $f(z)$ and $f_{\alpha\beta}(z)$ cannot both be integral functions, for if one is an integral function, the other has a branch at $z=1$. Hence the assertion holds good. So, the sequence $\theta = \begin{pmatrix} 0, & 0, & ...0 \\ 0, & 0, & ...0 \\ . \\ . \\ . \\ 0, & 0, & ...0 \end{pmatrix}$ belongs to both χ^2 and $\sigma^{\alpha\beta}\left(\chi^2\right)$. But this is the only sequence common to both these spaces. Hence $\chi^2 \bigcap \sigma^{\alpha\beta}\left(\chi^2\right) = \theta$.

3.2. Definition. Fix $m, n = 0, 1, 2, \cdots$. Given a sequence (x_{mn}), put $\xi_{m_p n_p} = \frac{\alpha_{1+m,1+n}+\alpha_{2+m,2+n}+\cdots+\alpha_{m+p,n+p}}{p(m+n)!}$ for $p = 1, 2, 3, \cdots$. Let $\left(\xi_{m_p n_p} : p = 1, 2, 3, \cdots\right) \in \chi^2$ uniformly in $m, n = 0, 1, 2, \cdots$. Then we call (x_{mn}) an "almost double gai sequence." The set of all almost double gai sequences is denoted by Δ^2.

3.11. Proposition. $\chi^2 \bigcap \sigma^{\alpha\beta}\left(\chi^2\right) = \Delta^2$, where Δ^2, is the set of all almost double gai sequences.

Proof. Put $m = 0, n = 0$. Then
$\left(\xi_{0p,0p}\right) \in \chi^2 \Leftrightarrow \left(\frac{\alpha_{11}+\alpha_{22}+\cdots+\alpha_{pp}}{p}\right) \in \chi^2$
$\Leftrightarrow \left|\alpha_{11}+\alpha_{22}+\cdots+\alpha_{pp}\right|^{1/m+n} \to 0\, as\, m, n\, and\, p \to \infty.$

$$\Leftrightarrow \alpha_{11}+\alpha_{22}+\cdots = 0 \tag{14}$$

$\Leftrightarrow (x_{mn}) \in cs_0^2$.
Therefore $\Delta \subset cs_0^2$
Put $m = 1, n = 1$. Then
$\left(\xi_{1p,1p}\right) \in \chi^2 \Leftrightarrow \left(\frac{\alpha_{22}+\cdots+\alpha_{pp}}{p}\right) \in \chi^2$
$\Leftrightarrow \left|\alpha_{22}+\cdots+\alpha_{pp}\right|^{1/m+n} \to 0 \,\text{as}\, m, n \,\text{and}\, p \to \infty.$

$$\Leftrightarrow \alpha_{22}+\alpha_{33}+\cdots = 0 \tag{15}$$

Similarly we get

$$\Leftrightarrow \alpha_{33}+\alpha_{44}+\cdots = 0 \tag{16}$$

$$\Leftrightarrow \alpha_{44}+\alpha_{55}+\cdots = 0 \tag{17}$$

and so on.
From (14) and (15) it follows that

$\alpha_{11} = (\alpha_{11} + \alpha_{22} + \cdots) - (\alpha_{22} + \alpha_{33} + \cdots) = 0.$
Similarly we obtain $\alpha_{22} = 0, \alpha_{33} = 0, \cdots$ and so on.

Hence $\Delta^2 = \theta$, where θ denots the sequence $\begin{pmatrix} 0, & 0, & ...0 \\ 0, & 0, & ...0 \\ . & & \\ . & & \\ . & & \\ 0, & 0, & ...0 \end{pmatrix}$.

Thus we have proved that $\chi^2 \bigcap \sigma^{\alpha\beta}\left(\chi^2\right) = \theta$ and $\Delta^2 = \theta$. Inotherwords, $\chi^2 \bigcap \sigma^{\alpha\beta}\left(\chi^2\right) = \Delta^2$. This completes the proof.

3.12. Proposition. $\chi^2_s = \chi^2 \bigcap cs^2_0$.

Proof. By Proposition 3.1 $\chi^2_s \subset \chi^2$. Also, since every double χ sequence ξ_{mn} is a double null sequence, it follows that (ξ_{mn}) is a double null sequence. Inotherwords $(\xi_{mn}) \in cs^2_0$. Thus $\chi^2_s \subset cs^2_0$. Consequently

$$\chi^2_s \subset \chi^2 \bigcap cs^2_0. \tag{18}$$

On the other hand, if $(\alpha_{mn}) \in \chi^2 \bigcap cs^2_0$, then $f(z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \alpha_{mn} z^{(m-1,n-1)}$ is an χ function. But $(\alpha_{mn}) \in cs^2_0$. So, $f(1) = \alpha_{11} + \alpha_{22} + \cdots = 0$. Hence $\frac{f(z)}{1-z} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} ((m+n)!\xi_{mn}) z^{(m-1,n-1)}$ is also an double gai funtion. Hence $(\xi_{mn}) \in \chi^2$. So $x = (x_{mn}) \in \chi^2_s$. But (x_{mn}) is arbitrary in $\chi^2 \bigcap cs^2_0$. Therefore

$$\chi^2 \bigcap cs^2_0 \subset \chi^2_s. \tag{19}$$

From (18) and (19) we get $\chi^2_s = \chi^2 \bigcap cs^2_0$. This completes the proof.

References

1. T.Apostol, Mathematical Analysis, Addison-wesley , London, 1978.
2. M.Basarir and O.Solancan, On some double sequence spaces, J. Indian Acad. Math., 21(2) (1999), 193-200.
3. T.J.I'A.Bromwich, An introduction to the theory of infinite series Macmillan and Co.Ltd. ,New York, (1965).
4. R.Colak and A.Turkmenoglu, The double sequence spaces $\ell^2_\infty(p), c^2_0(p)$ and $c^2(p)$, (to appear).
5. G.H.Hardy, On the convergence of certain multiple series, Proc. Camb. Phil. Soc., 19 (1917), 86-95.
6. F.Moricz, Extentions of the spaces c and c_0 from single to double sequences, Acta. Math. Hungerica, 57(1-2), (1991), 129-136.
7. F.Moricz and B.E.Rhoades, Almost convergence of double sequences and strong regularity of summability matrices, Math. Proc. Camb. Phil. Soc., 104, (1988), 283-294.

8. B.C.Tripathy, On statistically convergent double sequences, Tamkang J. Math., 34(3), (2003), 231-237.
9. A.Turkmenoglu, Matrix transformation between some classes of double sequences, Jour. Inst. of math. and Comp. Sci. (Math. Seri.), 12(1), (1999), 23-31.
10. P.K.Kamthan and M.Gupta, Sequence spaces and series, Lecture notes, Pure and Applied Mathematics, 65 Marcel Dekker, In c., New York , 1981.
11. A.Gökhan and R.Colak, The double sequence spaces $c_2^P(p)$ and $c_2^{PB}(p)$, Appl. Math. Comput., 157(2), (2004), 491-501.
12. A.Gökhan and R.Colak, Double sequence spaces ℓ_2^∞, ibid., 160(1), (2005), 147-153.
13. M.Zeltser, Investigation of Double Sequence Spaces by Soft and Hard Analitical Methods, Dissertationes Mathematicae Universitatis Tartuensis 25, Tartu University Press, Univ. of Tartu, Faculty of Mathematics and Computer Science, Tartu, 2001.
14. M.Mursaleen and O.H.H. Edely, Statistical convergence of double sequences, J. Math. Anal. Appl., 288(1), (2003), 223-231.
15. M.Mursaleen, Almost strongly regular matrices and a core theorem for double sequences, J. Math. Anal. Appl., 293(2), (2004), 523-531.
16. M.Mursaleen and O.H.H. Edely,Almost convergence and a core theorem for double sequences, J. Math. Anal. Appl., 293(2), (2004), 532-540.
17. B.Altay and F.Basar, Some new spaces of double sequences, J. Math. Anal. Appl., 309(1), (2005), 70-90.
18. F.Basar and Y.Sever, The space L_p of double sequences, Math. J. Okayama Univ, 51, (2009), 149-157.
19. N.Subramanian and U.K.Misra, The semi normed space defined by a double gai sequence of modulus function, Fasciculi Math., 46, (2010).
20. H.Kizmaz, On certain sequence spaces, Cand. Math. Bull., 24(2), (1981), 169-176.
21. A.Wilansky, Summability through Functional Analysis, North-Holland Mathematics studies, North-Holland Publishing, Amsterdam, Vol. 85 (1984).